Systems of Particles

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Based on Physics, 5th Ed. by Resnick, Halliday, Krane (Ch. 7)

1 The Motion of a Complex Object

- The motion of a complex object can be seperated into two components: the motion of the center of mass (translational) and the motion about the center of mass (rotational).

2 Many-particle Systems

- For a system of many particles, the center of mass vector $\vec{r}_{CM}$ is the average of the individual position vectors, weighted by their masses. That is, $\vec{r}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$
  - Note: In general, a weighted average is given by $\bar{x} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$
  - Time derivatives give $\vec{v}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{v}_i}{\sum_{i=1}^{n} m_i}$ and $\vec{a}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{a}_i}{\sum_{i=1}^{n} m_i}$
- r_{CM} can be split into $x_{CM} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$ and $y_{CM} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$
- Rearranging the expression for a_{CM} gives $\vec{a}_{CM} \sum_{i=1}^{n} m_i = \sum_{j=1}^{N} F_j$
  - Thus, the total force acting on a system of particles is given by the total mass of the particles, and the acceleration of the center of mass
- Since internal forces will cancel according to Newton's third law, this can be simplified to $\sum \vec{F}_{ext} = \vec{a}_{CM} \sum_{i=1}^{n} m_i$
  - That is, the translational motion of a system of particles can be analyzed using Newton's laws as if all the mass was concentrated in the center of mass.

3 Center of Mass of Solid Objects

- The location of the center of mass of a solid object is given by $\vec{r}_{CM} = \frac{1}{M} \int \vec{r} dm$

4 Conservation of Momentum in a System of Particles

- For a system of N particles, the total momentum of the system is given by $\vec{P} = \sum_{n=1}^{N} \vec{p}_n = M\vec{v}_{CM}$, where $M = \sum_{n=1}^{N} m_n$
  - Thus, $\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt}$
- In the center of mass frame, the total momentum os always zero (in systems with constant mass)

5 Systems of Variable Mass

- In a system with variable mass (rockets, etc.), $\vec{P}_i = M\vec{v}$ and $\vec{P}_f = (M + \Delta M)(\vec{v} + \Delta\vec{v}) + (-\Delta M)\vec{u}$, where ΔM is a negative quantity describing the mass lost, and $\vec{u}$ is the speed of the ejected mass
- Applying $\sum \vec{F}_{ext} = \lim_{\Delta t \to 0} \frac{\Delta \vec{P}}{\Delta t}$ gives $M\frac{d\vec{v}}{dt} = \sum \vec{F}_{ext} + \vec{v}_{rel}\frac{dM}{dt}$ where $\vec{v}_{rel} = \vec{u} - \vec{v}$ (velocity of ejected mass relative to object)
- Thrust - the force exerted on a rocket by the ejected gas. Represented by $\vec{v}_{rel}\frac{dM}{dt}$