# Wave Motion

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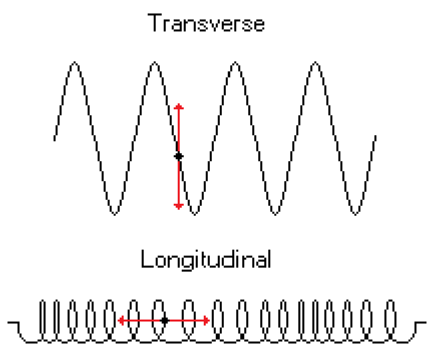
Based on Physics, 5th Ed. by Resnick, Halliday, Krane (Ch. 18)

## 1 Mechanical Waves

- Mechanical waves - waves that result from an oscillation of matter, such as sound waves and water waves
- In this chapter, we will concentrate on the oscillation of a stretched string
- The particles of matter in the medium of a mechanical wave do not experience a net displacement in the direction of the wave. They return to their equilibrium positions (more or less) once the wave has passed.

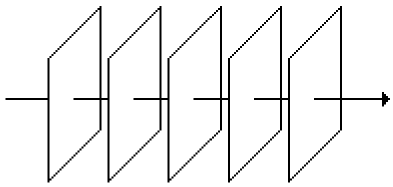
## 2 Types of Waves

- Transverse - describes a wave in which the motion of the particles is perpendicular. Includes waves on a string and light waves.
- Longitudinal - describes a wave in which the motion of the particles is perpendicular. Includes waves along a spring and sound waves.




- Some waves have both transverse and longitudinal elements.
- Waves can be one-dimensional (such as along a string or spring), two-dimensional (such as ripples on a pond's surface), or three-dimensional (such as sound or light emitted radially).

- Pulse - a single wave travelling through a medium. Each particle in the medium is at rest until the pulse reaches it. It then moves for a short time as the pulse passes through, before returning to rest.
- Train of waves - a continuous series of waves passing through a medium
- Wavefront - an area over which all points are in the same state of motion
- Ray - a line normal to the wavefronts of a wave, indicating the direction of propagation
- Plane wave - a three-dimensional wave in which conditions are the same everywhere on any plane perpendicular to the direction of propagation



# 3 Travelling Waves

- Ideal string - a string in which disturbances keep their form as they travel. Requires energy dissipation to be negligible
- Consider an arbitrary wave pulse with a shape given by $f(x)$ and speed $v$. If the y-coordinate of the waveform is given by $y(x, 0) = f(x)$ at time $t = 0$, it is given at any time by $y(x, t) = f(x - vt)$
  - Think back to algebra - this is a simple horizontal translation by the distance the wave has travelled ($vt$)
- Phase (of a wave) - a certain point along a wave
- For any given phase, $x - vt = constant$ (otherwise the wave would change shape as it moves)
- Phase velocity - the velocity of any given point on a wave (i.e. the velocity of the "whole" wave)
- If the wave moves in the negative $x$ direction, $x + vt$ is the conserved quantity
- Sinusoidal Waves
  - Harmonic wave - a wave for which $f(x-vt)$ is a sine or cosine function
  - Amplitude ($y_m$) - In harmonic waves, the maximum transverse (vertical) displacement
  - Wavelength ($\lambda$) - The distance between two adjacent identical points on a wave (ex: peak to peak)

- The equation of a harmonic wave travelling in the $+x$ direction with phase speed $v$ is given by $y(x,t) = y_m \sin[\frac{2\pi}{\lambda}(x - vt)]$
  - Note that $y(x,t) = y(x + n\lambda, t)$ for any integer $n$
- Period ($T$) - The time necessary for a point at any particular $x$ coordinate to undergo one cycle of transverse motion. Given by $T = \frac{\lambda}{v}$
- Frequency ($f$) - The inverse of the period. $f = \frac{1}{T}$
- The equation for a harmonic wave can also be written as $y(x,t) = y_m \sin[2\pi(\frac{x}{\lambda} \cdot \frac{t}{T})]$. Using the definitions of wave number $k = \frac{2\pi}{\lambda}$ and $\omega = \frac{2\pi}{T} = 2\pi f$, we can write this as $y(x,t) = y_m \sin(kx - \omega t)$
- The phase speed (aka wave speed) of a harmonic wave is given by $v = \lambda f = \frac{\lambda}{T} = \frac{\omega}{k}$

* Transverse Velocity of a Particle
  - The transverse velocity of any given particle on the wave is given by $u_y(x,t) = \frac{\partial y}{\partial t} = -y_m \omega \cos(kx - \omega t)$
    - So $u_y$ ranges from $-y_m\omega$ to $y_m\omega$
  - The transverse acceleration is given by $a_y(x,t) = \frac{\partial^2 y}{\partial t^2} = -y_m \omega^2 \sin(kx - \omega t) = -\omega^2 y$
    - Note that this has the same form as the acceleration of a particle undergoing simple harmonic motion - that is, each particle in a harmonic wave is undergoing simple harmonic motion in the transverse direction.
* Phase and Phase Constant
  - The equation of a harmonic wave that does not necessarily have a transverse displacement of 0 at $x = 0$ and $t = 0$ is given by $y(x,t) = y_m \sin(kx - \omega t - \phi)$, where $\phi$ is an angle known as the phase constant
    - The argument of the sine function, $(kx - \omega t - \phi)$, is known as the phase of the wave
  - Two waves with the same phase, or with phases differing by an integer multiple of $2\pi$, are said to be "in phase"
  - If we rewrite $y(x,t)$ as $y(x,t) = y_m \sin[k(x - \frac{\phi}{k}) - \omega t]$, we can see that a wave with phase constant $\phi$ will be shifted to the right by $\frac{\phi}{k}$ from an otherwise-equivalent wave with phase constant 0.
  - If we rewrite $y(x,t)$ as $y(x,t) = y_m \sin[kx - \omega(t + \frac{\phi}{\omega})]$, we can see that a certain point on a wave with phase constant $\phi$ will reach a certain transverse displacement by an amount $\frac{\phi}{\omega}$ sooner than an otherwise equivalent wave with phase constant 0.
  - When one wave is ahead of another in time or space, it is said to "lead"; when it is behind, it is said to "lag"

## 4 Wave Speed on a Stretched String

- On an ideal string, the phase speed is independent of frequency and wavelength, depending instead on the properties of the medium
- Dispersive - describes a medium for which the phase speed depends on frequency and wavelength
- The speed of a wave on an ideal string is given by $v = \sqrt{\frac{F}{\mu}}$, where $F$ is the tension in the string and $\mu$ is the string's linear mass density
- When a wave passes from one medium to another (for example, two strings of different linear mass densities tied together), the frequency remains the same (otherwise there would be a discontinuity in the wave), but the wavelengths are different
  - Since $f_1 = f_2$, the wavelengths and velocities of the wave in the two mediums are given by $\frac{v_1}{\lambda_1} = \frac{v_2}{\lambda_2}$

## 5 Energy in Wave Motion

- The rate at which kinetic energy is transported along a wave is given by $\frac{dK}{dt} = \frac{1}{2}\mu\omega^2 y_m^2 v \cos(kx - \omega t)$
  - From taking the time derivative of $KE = \frac{1}{2}mv^2$ for an infinitesimal length of string, $dm = \mu dx$
- The rate at which potential energy is transported along a wave is given by $\frac{dU}{dt} = \frac{1}{2}Fvy_m^2 k^2 \cos^2(kx - \omega t)$
  - From taking the time derivative of $U = F\Delta x$ for an infinitesimal length of string stretching from length $dx$ to length $dl$
- $dK$ and $dU$ are both 0 when the element has maximum displacement, and have their maximum values when the element crosses the x-axis
  - The total mechanical energy of the mass element is not constant because it is not an islated system (work is done on it by neighboring elements)
- By substituting $F = v^2\mu = (\frac{\omega}{k})^2\mu$ into $\frac{dU}{dt}$, we can see that $\frac{dU}{dt} = \frac{dK}{dt}$
- The rate at which mechanical energy is transmitted along the string, that is, the power, is given by $P = \frac{dE}{dt} = \frac{dK}{dt} + \frac{dU}{dt} = \mu\omega^2 y_m^2 v \cos^2(kx - \omega t)$
- The average value of $\cos^2(kx - \omega t)$ over any number of full cycles is $\frac{1}{2}$, so for long observation periods we can say $P_{av} = \frac{1}{2}\mu\omega^2 y_m^2 v$
- Intensity (of a spherical wave) - the average power per unit area transmitted across an area $A$, perpendicular to the direction the wave is travelling in. Given by $I = \frac{P_{av}}{A}$. Unit: $\frac{W}{m^2}$

# 6 The Principle of Superposition

- Principle of superposition (regarding waves) - When several waves combine at a point, the displacement $y(x,t)$ of any particle at any given time is the sum of the displacements that each individual wave acting alone would give it. That is, $y(x,t) = y_1(x,t) + y_2(x,t) + \ldots$ where $y_1$, $y_2$, $\ldots$ are individual wave displacements

# 7 Interference of Waves

- Interference - when two or more waves combine at a particular point
- A point on a string is said to be undergoing "constructive interference" at any point for which the total amplitude is greater than that of any of the individual waves, and "destructive interference" if it is less than that of any of the individual waves
- Consider two otherwise-identical waves with different phase constants, $y_1(x,t) = y_m \sin(kx - \omega t - \phi_1)$ and $y_2(x,t) = y_m \sin(kx - \omega t - \phi_2)$. Applying a trig identity, we can find the total amplitude $y(x,t) = 2y_m \cos(\frac{\Delta\phi}{2} \sin(kx - \omega t - \phi\prime)$, where $\phi\prime = \frac{\phi_1 + \phi_2}{2}$. $\Delta\phi$ is a quantity called the phase difference and is given by $\Delta\phi = \phi_2 - \phi_2$
  - Note that the resultant wave has an amplitude of $2y_m \cos(\frac{\Delta\phi}{2})$. When $\Delta\phi = 0$, the waves are exactly in phase, and the resultant wave has its maximum amplitude, $2y_m$. When $\Delta\phi = 180$, its amplitude is 0.

# 8 Standing Waves

- When two component waves of equal amplitude and frequency move in ¡u¿opposite¡/u¿ directions along a string, we observe a phenomenon called a "standing wave"
- Node (of a standing wave) - certain points along the string which have 0 displacement at all times
- Antinode (of a standing wave) - certain points along the string, halfway between the nodes, where the displacement oscillates with the largest amplitude
- We can find the equation for a standing wave by adding $y_1(x,t) = y_m \sin(kx - \omega t)$ and $y_2(x,t) = y_m \sin(kx + \omega t)$ and apllying a trig identity to get $y(x,t) = 2y_m \sin(kx)\cos(\omega t)$
  - Note that the particle at any $x$ has the same angular frequency $\omega$, but has an amplitude that depends on its location, $2y_m \sin(kx)$.
  - The maximum amplitude, $2y_m$, occurs where $kx = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$ which can be written as $x = (n + \frac{1}{2})\frac{\lambda}{2}$ where $n = 0, 1, 2, \ldots$ (positions of antinodes)
  - The minimum amplitude, 0, occurs where $kx = 0, \pi, 2\pi, \ldots$ which can be written as $x = n\frac{\lambda}{2}$ where $n = 0, 1, 2, \ldots$ (positions of nodes)

  - Looking at the equation $y(x,t)$, we can see that each point on a standing wave is undergoing simple harmonic motion in the vertical direction. Thus, each point has maximum (100%) kinetic energy at $y = 0$, and maximum (100%) potential energy at its farthest point from $y = 0$ ($y_m$)

- On reflection from a fixed end, a transverse wave undergoes a phase change of 180.

  - In the case of a string attached to a wall, this occurs because when the wave reaches the wall, it exerts a force upon it which the wall then exerts in the opposite direction on the string (Newton's third law)

- At a free end, a transverse wave is reflected without change of phase