Oscillations

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Based on Physics, 5th Ed. by Resnick, Halliday, Krane (Ch. 17)

1 Oscillating Systems

- Amplitude - in an oscillating system, the maximum displacement from the equilibrium position
- Period (T) - the time necessary to complete one cycle of oscillatory motion (unit: seconds)
- Frequency (f) - the number of cycles completed per unit time (unit: $\frac{1}{s}$ or Hz)
- Frequency and period are related by $f = \frac{1}{T}$
- Two equivalent ways of describing the conditions for oscillation:
  - The force must always act to restore the particle to equilibrium
  - The potential energy must have a minimum at the equilibrium position

2 The Simple Harmonic Oscillator

- Harmonic - describes oscillations that can be described in terms of sine and cosine functions
- Simple harmonic oscillator - an oscillating one-dimensional system consisting of a particle subject to a force $F_x(x) = -kx$, where k is a constant and x is the displacement of the particle from its equilibrium position
- Simple harmonic motion - the motion undergone by a particle in a simple harmonic oscillator
- The potential energy of a particle in simple harmonic motion is given by $U(x) = \frac{1}{2}kx^2$ (from $F_x(x) = -\frac{dU}{dx}$)
- An example of a simple harmonic oscillator is an object of mass m attached to a spring of spring constant k that is free to move over a frictionless surface
- Applying Newton's second law and $a_x = \frac{d^2x}{dt^2}$ to $F_x(x) = -kx$ gives $\frac{d^2x}{dt^2} = -\frac{k}{m}x$. This is known as the "equation of motion" of the simple harmonic oscillator.

3 Simple Harmonic Motion

- One solution to the equation of motion is $x = x_m \cos(\omega t + \phi)$, where $\omega = \sqrt{\frac{k}{m}}$ and x_m and ϕ are arbitrary constants which, in practice, can be determined by the initial conditions of the system.
  - Physically, x_m is the amplitude (maximum displacement) and ϕ is known as the "phase constant", which represents how far apart two wave crests are in two otherwise-equivalent waves.
  - The quantity $(\omega t + \phi)$ is known as the phase of the motion
  - This solution repeats itself after a time $\frac{2\pi}{\omega}$, so the period is $T = \frac{2\pi}{\omega}$
  - The frequency of a simple harmonic motion is independent of the amplitude
- Once motion has started, x_m, ϕ, and f remain constant (unless a force disturbs the system)
- We can find the velocity and acceleration of an object in simple harmonic motion by taking the time derivatives of x.
  - $v_x = \frac{dx}{dt} = -\omega x_m \sin(\omega t + \phi)$
  - $a_x = \frac{d^2x}{dt^2} = -\omega^2 x_m \cos(\omega t + \phi)$

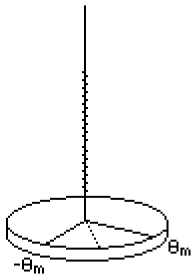
4 Energy in Simple Harmonic Motion

- We can get expressions for the potential and kinetic energies of a simple harmonic oscillator by plugging in x and v_x into $U = \frac{1}{2}kx^2$ and $K = \frac{1}{2}mv^2$:
  - $U = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi)$
  - $K = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi) = \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi)$
- Adding these together and applying a trig identity shows that the total energy is constant: $E = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi) + \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi) = \frac{1}{2}kx_m^2[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)] = \frac{1}{2}kx_m^2$
- Speed is maximum @ $x = 0$ and $x = \pm x_m$

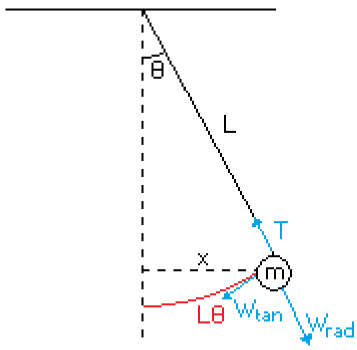
5 Applications of Simple Harmonic Motion

- The Torsional Oscillator
  - The equation for angular simple harmonic motion (such as a disk that is free to spin on a twisted wire from $-\theta_m$ to θ_m) is $\tau_z = -\kappa\theta$, where κ is the "torsional constant" of the wire
  - Combining this with $\sum \tau_z = I\alpha_z = I\frac{d^2\theta}{dt^2}$, we can write the equation for angular simple harmonic motion: $\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta$. It is mathematically identical to the equation for linear simple harmonic motion.

- The solution is $\theta = \theta_m \cos(\omega t + \phi)$, where $\omega = \sqrt{\frac{\kappa}{I}}$ and ¡u¿not¡/u¿ $\frac{d\theta}{dt}$
- The period is given by $T = 2\pi\sqrt{\frac{I}{\kappa}}$

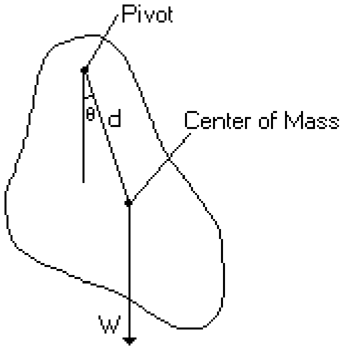


- The Simple Pendulum
  - The period of a simple pendulum is given by $T = 2\pi\sqrt{\frac{L}{g}}$
    * From the fact that the restoring force is the tangential component of the gravitational force, $W_{tan} = -mg\sin\theta$, and applying the small angle approximation $\sin\theta \approx \theta$, which also implies $x \approx L\theta$, so $W_{tan} = -mg\frac{x}{L}$. Thus, $k = \frac{mg}{L}$ for this system, and $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{mL}{mg}} = 2\pi\sqrt{\frac{L}{g}}$



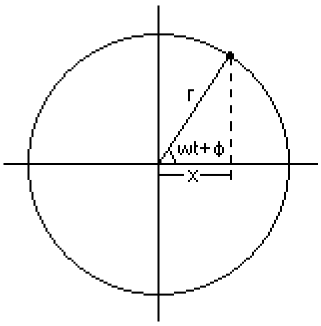
- The Physical Pendulum
  - Physical pendulum - a rigid body mounted so that it can swing in a vertical plane about some axis passing through it
    * From $\tau_z = -Mgd\sin\theta \approx -Mgd\theta$ where M is the mass of the body, thus $\kappa = Mgd$ and $T = 2\pi\sqrt{\frac{I}{Mgd}}$
    * Note: Solving for I is a convenient way to determine I for irregularly-shaped objects
    * A physical pendulum has the same period as a simple pendulum with $L = \frac{I}{Md}$. This point on the physical pendulum is known as the center of oscillation (or center of percussion). A force

delivered at this point will not be felt at the pivot point.



6 Simple Harmonic Motion and Uniform Circular Motion

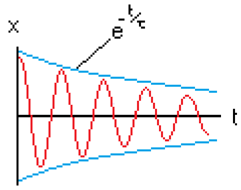
- Simple harmonic motion can be described as the projection of uniform circular motion along a diameter of the circle (think of a side-on view of a moon orbiting a planet)
  - This becomes apparent if we consider the x-component of a particle undergoing uniform circular motion. At angle $\omega t + \phi$, its horizontal position is given by $x(t) = r\cos(\omega t + \phi)$



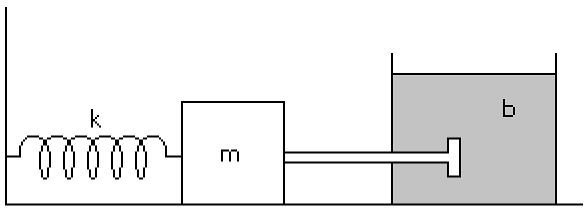
7 Damped Harmonic Motion

- Damping - a gradual loss of amplitude in an oscillating system
  - Causes of damping include friction, air resistance, and internal forces
- In many cases, the decrease in amplitude due to damping can be accounted for by multiplying $x(t)$ by an exponential function $e^{-\frac{t}{\tau}}$, where τ is known as the "damping time constant" or "mean lifetime", and is equal to the time it takes for the amplitude to drop to $\frac{1}{e}$ of its original value.
  - Peaks in $x(t)$ correspond to times when $\cos(\omega t + \phi) = 1$. These also correspond to turning points of the motion, so the energy is all kinetic

and $E = \frac{1}{2}kx_m^2 e^{-\frac{2t}{\tau}}$. Note that energy decreases twice as rapidly as amplitude.



- One example of a source of damping is a frictionless mass-and-spring system in which the mass is attached to a plug in a viscous liquid. The damping force can be represented in the same way as a drag force, $F_x = -bv_x$, where b is a positive constant that depends on the properties of the fluid and shape of the plug. By Newton's second law, $-kx - bv_x = ma_x$, or $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$. The solution to this differential equation is $x(t) = x_m e^{-\frac{bt}{2m}} \cos(\omega' t + \phi)$, where $\omega' = \sqrt{\frac{k}{m} - (\frac{b}{2m})^2}$ and b is small.



- Critical Damping - the case in which $b = 2\sqrt{km}$, causing ω' to be 0 and the position to immediately start exponentially decaying to 0.

- Sufficiently weak damping does not significantly change ω

8 Forced Oscillations and Resonance

- Natural frequency - the frequency an oscillating system will have if left on its own. Generally determined by $\sqrt{\frac{k}{m}}$

- Forced oscillations - an oscillator to which an external sinusoidal force is applied. The motion has the frequency of the applied force rather than the natural frequency of the oscillator. The amplitude depends on the relationship between the natural frequency and the frequency of the applied force.

- When considering forced oscillations, the system is usually assumed to have a damping force (otherwise, the energy from the external force would accumulate, and the amplitude would increase without bound)

- For an applied force $F_x(t) = F_m \sin(\omega'' t)$, the closer ω" is to the oscillator's natural frequency ω, the higher the amplitude of the motion

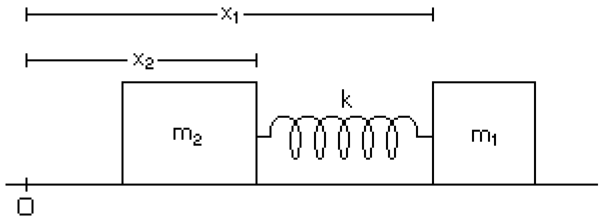
- Resonance - describes a system in which the frequency of the applied force is equal to the oscillator's natural frequency

- At resonance, the system oscillates at the frequency of the driving force, at constant amplitude

- For a system subject to damping force $-bv_x$ and driving force $F_m \cos(\omega'' t)$, we can write $-kx - bv_x + F_m \cos(\omega'' t) = ma_x$, or $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_m \cos(\omega'' t)$. The solution to this equation is $x(t) = \frac{F_m}{G}\cos(\omega'' t - \beta)$, where $G = \sqrt{m^2(\omega''^2 - \omega^2)^2 + b^2\omega''^2}$ and $\beta = \cos^{-1}(\frac{b\omega''}{G})$. Thus, the amplitude of this motion is $\frac{F_m}{G}$.

9 Two-body Oscillations

- When analyzing the motion of two masses connected to a spring, it is simpler to use center-of-mass and relative seperation components rather than the distances x_1 and x_2. Furthermore, the center-of-mass motion is generally not of interest and can be ignored.



- The length of the spring at any given time is $x_1 - x_2$ (where x_1 is the position of the mass farther from the origin). If the equilibrium length of the spring is L, we can write the change in length as $x = x_1 - x_2 - L$.

- The force on each mass by the spring has magnitude $F = kx$, each exerted in opposite directions

- By Newton's second law, $-kx = m_1\frac{d^2x_1}{dt^2}$ and $-kx = m_2\frac{d^2x_2}{dt^2}$. Multiplying the first equation by m_2 and the second by m_1 and combining them gives $\frac{m_1m_2}{m_1+m_2}\frac{d^2}{dt^2}(x_1-x_2) = -kx$. The quantity $\frac{m_1m_2}{m_1+m_2}$ is known as the "reduced mass" m. Since $\frac{d}{dt}(x_1-x_2)$ gives the same result as $\frac{d}{dt}(x_1-x_2-L) = \frac{dx}{dt}$, we can write $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$, which is mathematically equivalent to the formula describing a single oscillating mass.

- The expressions for $x(t)$, $v_x(t)$, and $a_x(t)$ are identical to those of a single oscillating mass, but keep in mind that for a system of two masses, these refer to relative quantities $x_1 - x_2$, $v_{x1} - v_{x2}$, $a_{x1} - a_{x2}$