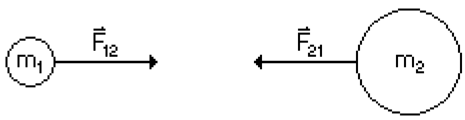
Gravitation

Notes prepared by Megan L. Barry

Based on Physics, 5th Ed. by Resnick, Halliday, Krane (Ch. 14)

1 Newton's Law of Universal Gravitation

- Newton's Law of Universal Gravitation: Every particle in the universe attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the particles.
- Gravitational constant - a universal constant that can be thought of as representing the "strength" of the gravitational force. Given by $G = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$
- The magnitude of the gravitational force between two objects with masses m_1 and m_2 a distance r apart is given by $F = G\frac{m_1 m_2}{r^2}$
- Unit vector - a dimensionless vector with a length of 1. Used to specify a direction. In gravitation, we use the unit vectors $\hat{\vec{r}}_{12}$ and $\hat{\vec{r}}_{21}$ to denote a vector pointing toward m_1 from m_2 and a vector pointing toward m_2 from m_1, respectively. We can write these as $\hat{\vec{r}}_{12} = \frac{\vec{r}_{12}}{r_{12}}$ and $\hat{\vec{r}}_{21} = \frac{\vec{r}_{21}}{r_{21}}$. (note: $|\hat{\vec{r}}_{12}| = |\hat{\vec{r}}_{21}| = 1$)
- As a vector, the gravitational force is given by $\vec{F}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\hat{\vec{r}}_{12}$ (force on m_1 due to m_2) and $\vec{F}_{21} = -G\frac{m_1 m_2}{r_{21}^2}\hat{\vec{r}}_{21}$.



- Of course, gravitational forces form action-reaction pairs ($\vec{F}_{12} = -\vec{F}_{21}$)
- The total graitational force acting on a body is the vector sum of all of the individual gravitational forces acting upon it.
- If the radii of the objects is very small compared to the distance between them, the objects can be treated as particles in most cases.

2 Gravitation Near the Earth's Surface

- Combining $F = G\frac{m_1 m_2}{r^2}$ and $F = mg$ gives an expression for the free-fall acceleration at the surface of a spherical, non-rotating massive object with a spherically symmetric mass distribution: $g = \frac{GM}{r^2}$
  - This expression is still an acceptable approximation for many massive objects that don't quite fit these criteria (i.e. the Earth)

3 The Two Shell Theorems

- Shell Theorem 1: A uniformly dense spherical shell attracts an external particle as if all the mass of the shell were concentrated at its center.
  - Thus, a spherically symmetric body attracts objects as if all its mass were concentrated at the center.
- Shell Theorem 2: A uniformly dense spherical shell exerts no gravitational force on a particle located anywhere inside of it.

4 Gravitational Potential Energy

- Gravitational force is conservative.
  - The work done in moving a particle between two points in a gravitational field depends only on the radial distance it moved (otherwise $F \perp d$), so the work done is path-independent

The work done in moving a particle of mass m from point a to point b in the presence of a mass M at the origin is given by $W_{ab} = \int_a^b \vec{F} \cdot d\vec{r} = -\int_a^b F dr = -\int_{r_a}^{r_b} \frac{GMm}{r^2} dr = GMm(\frac{1}{r_b} - \frac{1}{r_a})$. Thus, $\Delta U = -W_{ab} = GMm(\frac{1}{r_b} - \frac{1}{r_a})$

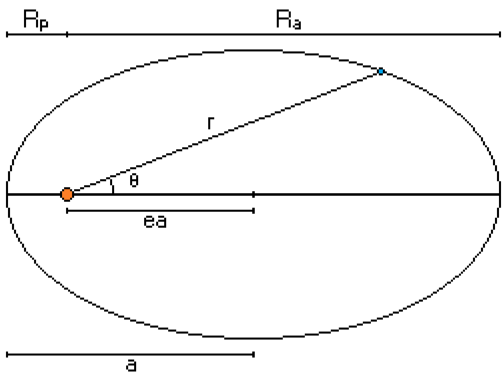
- If we define $r = \inf$ as having $U = 0$, the potential energy at a point can be expressed as $U(r) = -\frac{GMm}{r}$
  - Minimum U is 0, and U gets more negative with increasing r
  - Potential energy is a property of a system, not a single object. But we can get away with saying an object near Earth's surface "has" potential energy because the resulting kinetic energy of the object is much, much larger than the resulting kinetic energy of the Earth.
- Escape speed - the minimum initial speed required for a projectile to "escape" a nearby massive object's gravitational pull. Specifically, the initial speed at which its speed decreases gradually to zero just as its distance approaches infinity.
  - A projectile in this situation has $K_i = \frac{1}{2}mv^2$ and $U_i = -\frac{GMm}{R}$, and $K_f = 0$ and $U_f = 0$, where R is the radius of the massive object. Solving for v gives $v_{esc} = \sqrt{\frac{2GM}{R}}$

        * Rotation of the massive object can slightly change this value (with v_T)

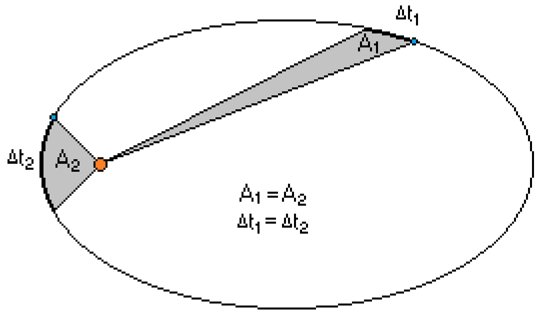
- Another way to interpret $W_{ab} = -\Delta U$: The potential energy of a system of particles is equal to the work done by an external agent to assemble the system, starting from infinite separation
- Binding energy - the energy required to move a system of particles to infinite separation

5 The Motions of Planets and Satellites

- For now, when considering orbital motion, we make the following simplifications:
    - Consider the gravitational force only between the orbiting body and the central body (ignore perturbations from nearby objects)
    - Assume the central body is much more massive than the orbiting body (ignore the central body's motion about the center of mass)
- Kepler's Laws of Planetary Motion
    - Law of Orbits: Planets move in elliptical orbits with the Sun at one focus.



        * Eccentricity e - the "flatness" of an ellipse; has a value between 0 and 1 (e=0 corresponds to a circle)
        * Perihelion - the minimum distance from the Sun (general term - periapsis) $R_p = a(1 - e)$
        * Aphelion - the maximum distance from the Sun (general term - apoapsis) $R_a = a(1 + e)$
    - Law of Areas: A line joining any planet to the Sun sweeps out equal areas in equal times.



        * This law is equivalent to the law of conservation of momentum.
            · Estimating the swept-out area as a triangle, $\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{\frac{1}{2} r(r\Delta\theta}{\Delta t} = \frac{1}{2} r^2 \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega$. For a rotating particle, $L_z = mr^2\omega$. So $\frac{dA}{dt} = \frac{L_z}{2m}$. Since L_z and m are constant, $\frac{dA}{dt}$ is constant.
    - Law of Periods: The square of the period (T) of any planet about the Sun is proportional to the cube of the planet's semimajor axis. That is, $T^2 \propto a^3$ or $\frac{T^2}{a^3}$ = a constant.
        * Let's use $F = ma$ to find the exact relationship between T and r for circular orbits. $F = ma => \frac{GMm}{r^2} = m\frac{v^2}{r} => \frac{GM}{r} = v^2 = (\frac{2\pi r}{T})^2$. Solving for T^2 gives $T^2 = (\frac{4\pi^2}{GM}) r^3$.

- The kinetic energy of an orbiting body can be found by combining $K = \frac{1}{2} m\omega^2 r^2$ with $\frac{GMm}{r^2} = m\frac{v^2}{r}$ to get $K = \frac{GMm}{2r}$
- The total energy of a circular orbit is $E = -\frac{GMm}{2r}$
    - From $E = K + U = \frac{GMm}{2r} + (-\frac{GMm}{r})$
    - E is constant and negative
    - It can be shown that this expression is also valid for elliptical orbits (replacing r with a)
    - If K is increased such that $E =$), the orbit would become parabolic. For $E > 0$, it would become hyperbolic.

6 The Gravitational Field

- Instead of considering gravity as a force-at-a-distance, we can consider a mass distribution as setting up a "gravitational field" where every point in space has an associated value $\vec{g} = \frac{\vec{F}}{m_0}$, which gives the gravitational force per unit mass that an object would experience at that point in space.
- vector field - a vector quantity that has a certain value for every point in space (ex: gravitational field)
- scalar field - a scalar quantity that has a certain value for every point in space (ex: temperature)

7 Modern Developments in Gravitation

- Stars in the outer regions of galaxies have orbital velocities much higher than we would expect based on the central mass of starrs they orbit around. This suggests the presence of an unidentified form of matter present in galaxies, which physicists have decided to call "dark matter".
- Principle of Equivalence: For an observer n an isolated chamber, it is impossible to determine whether they are on Earth's surface or traveling through empty space at a constant acceleration of $9.8\frac{m}{s^2}$.