Angular Momentum

Notes prepared by Megan L. Barry

Based on Physics, 5th Ed. by Resnick, Halliday, Krane (Ch. 10)

1 Angular Momentum of a Particle

- The angular momentum of a particle is given by $\vec{l} = \vec{r} \times \vec{p}$, where $\vec{r}$ is the position vector and $\vec{p}$ is the particle's linear momentum
  - Magnitude: $l = rp \sin\theta$, where θ is the smaller angle between r and p (direction given by right hand rule)
  - Units: $\frac{kg \cdot m^2}{s}$
- The net torque acting on a particle is equal to the rate of change of its angular momentum. That is, $\sum \vec{\tau} = \frac{d\vec{l}}{dt}$
  - This relation can be derived by taking the time derivative of $\vec{l} = \vec{r} \times \vec{p}$

2 Systems of Particles

- The total angular momentum of a system of particles is given by $\vec{L} = \vec{l}_1 + \vec{l}_2 + \ldots$
- By Newton's third law, the internal torque of a system of particles is zero, so $\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$. That is, the net external torque on a system of particles is equal to the rate of change of the system's total angular momentum.
  - Analogous to $\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt}$. A force changes linear momentum, a torque changes angular momentum.
- When a torque is applied to an object, the component of $\vec{\tau}$ parallel to $\vec{L}$ changes the magnitude of $\vec{L}$ but not the direction. The component of $\vec{\tau}$ perpendicular to $\vec{L}$ changes the direction of $\vec{L}$ but not its magnitude.
- $\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$ is true in an inertial reference frame if the origin is chosen to be the center of mass of the object of interest.

3 Angular Momentum and Angular Velocity

- Wobbling occurs when angular velocity is not parallel to the angular momentum (i.e. when a body is not axially symmetric).

- The value of $\vec{l}$ is dependent on the chosen origin point.
- The z component of angular momentum is given by $l_z = l \sin\theta = rp\sin\theta = rmv\sin\theta = rm(r\sin\theta) = m(r\sin\theta)^2\omega$. Thus, $l_z = I\omega$.
- For rigid bodies with axial symmetry, $\vec{L} = I\vec{\omega}$. For any other rigid body, $\vec{L}_z = I\vec{\omega}$.
- As a particle revolves in a circle about an axis, $\vec{l}_z$ is constant, but the perpendicular component of $\vec{l}$ changes direction.

4 Conservation of Angular Mometum

- Law of conservation of angular momentum: If the net external torque acting on a system is zero, the total vector angular momentum of a system remains constant. That is, $\vec{L}_i = \vec{L}_f$
- For a rigid body rotating about the z-axis, the z-component of $\vec{L}$ is also constant. That is, $L_{z,i} = L_{z,f}$, or $I_i\omega_i = I_f\omega_f$
- Angular momentum has a stabilizing effect on an object's axis of rotation. The higher the angular momentum, the less a given applied torque can change the object's axis of rotation. $\theta = \arctan\frac{\Delta L_\perp}{L}$

5 The Spinning Top

- Precession - when an object's axis of rotation slowly moves about the vertical axis (an external torque changes the direction but not the magnitude of the angular momentum)
- An object undergoing precession has an angular speed of precession $\omega_P = \frac{d\phi}{dt}$. Its relation to the external torque is $\vec{\tau}_{ext} = \vec{\omega}_P \times \vec{L}$
- An object undergoing precession can be considered to have two components of angular momentum: rotational angular momentum (about its symmetry axis) and precessional angular momentum
- nutation - a slight wobble of the rotational axis around the precessional circle